Internal Consistency of Fault-Tolerant Quantum Error Correction in Light of Rigorous Derivations of the Quantum Markovian Limit

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We critically examine the internal consistency of a set of minimal assumptions entering the theory of fault-tolerant quantum error correction for Markovian noise. These assumptions are: fast gates, a constant supply of fresh and cold ancillas, and a Markovian bath. We point out that these assumptions may not be mutually consistent in light of rigorous formulations of the Markovian approximation. Namely, Markovian dynamics requires either the singular coupling limit (high temperature), or the weak coupling limit (weak system-bath interaction). The former is incompatible with the assumption of a constant and fresh supply of cold ancillas, while the latter is inconsistent with fast gates. We discuss ways to resolve these inconsistencies. As part of our discussion we derive, in the weak coupling limit, a new master equation for a system subject to periodic driving.

I. INTRODUCTION

The theory of fault-tolerant quantum error correction (FT-QEC) is one of the pillars that the field of quantum information rests on. Starting with the discovery of quantum error correcting codes [1, 2], and the subsequent introduction of fault tolerance [3], this theory has been the subject of many improvements and important progress [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], leading to the well-known error correction threshold condition. Most recently, work by Steane [14] and Knill [17] (see also Reichardt [16]) has pushed the threshold down to values that are claimed to be very close to being within experimental reach. A notable feature of much of the work on FT-QEC is that the error models are *phenomenological*. By this we mean that the underlying models often do not start from a Hamiltonian, microscopic description of the system-bath interaction, but rather from a higher level, effective description, most notably that of Markovian dynamics. E.g., Knill writes: "We assume that a gate's error consists of *random, independent* applications of products of Pauli operators with probabilities determined by the gate" (our emphasis) [17]. This approach is natural given the considerable difficulty of obtaining error thresholds starting from a purely Hamiltonian description. Nevertheless, Hamiltonian approaches to decoherence management in a fault-tolerant setting have been pursued, e.g., a mixed phenomenological-Hamiltonian treatment of FT-QEC [18, 19, 20, 21], and a fully Hamiltonian study of fault tolerance in dynamical decoupling [22]. Also noteworthy are recent mixed phenomenological-Hamiltonian *continuous time* treatments of QEC [23, 24, 25].

Here we are concerned with a critical re-evaluation of the physical assumptions entering the theory of FT-QEC. We scrutinize, in particular, the consistency of the assumption of Markovian dynamics within the larger framework of FT-QEC. We point out that there may be an inherent inconsistency in the theory of Markovian FT-QEC, when viewed from the perspective of the validity of the Markovian approximation. We begin by briefly reviewing, in Section II, a set of minimal and standard, universally agreed upon assumptions made in Markovian FT-QEC theory. We then review, in Section III, the derivation of Markovian master equations, emphasizing the physical assumptions entering the Markovian approximation, in particular the requirement of consistency with thermodynamics. Having delineated the set of assumptions entering FT-QEC and the quantum Markov approximation, we discuss in Section IV the internal consistency of Markovian FT-QEC theory. We point out where according to our analysis there is an inconsistency, and discuss possible objections. In Section VI we then discuss how one may overcome the inconsistency using a variety of alternative approaches, including adiabatic quantum computing (QC), holonomic QC, topological QC, and recent work on FT-QEC in a non-Markovian setting [19, 20, 21]. We conclude in Section VII.

II. REVIEW OF STANDARD ASSUMPTIONS OF FT-QEC

The following are a set of minimal assumptions made in the theory of FT-QEC [3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17]:

1. **A1** — *Gates can be executed in a time τ_g such that $\tau_g \omega = O(\pi)$, where ω is a typical Bohr or Rabi frequency.*[1]

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[1] One might object that slower (even adiabatic) gates could be used instead. We analyze this possibility in detail in Section V D, and show that it does not lead to an improvement.

2. **A2** — *A constant supply of fresh and nearly pure ancillas*: at every time step we are given a supply of many qubits in the state $|0\rangle$, each of which can be faulty with some error parameter $\eta \ll 1$.

3. **A3** — *Error correlations decay exponentially in time and space.*

Some remarks:
(i) **A1** is not typically stated explicitly in the FT-QEC literature, but can be understood as resulting from the definition of a quantum gate, which is a unitary transformation $U = \exp(iA)$; when $A = \tau_g H$, where H is a Hamiltonian generating the gate, **A1** follows from the absence of a free parameter: when τ_g is scaled up H (and hence its eigenvalues) must be scaled down, and *vice versa.*
(ii) The distinction between Bohr and Rabi frequencies in **A1** is related to the application of constant *vs* periodic control fields, respectively. In the case of a constant control field **A1** can be understood as the condition that saturates the "Margolus-Levitin theorem" [26], which states that the time required to transform an initial state $|\psi\rangle$ to an orthogonal state $|\psi^\perp\rangle$ using a constant Hamiltonian H is lower-bounded by $\tau_{\min} = \pi\hbar/(2E)$, where $E = \langle\psi|H|\psi\rangle$; when $|\psi\rangle$ is an eigenstate of H we have $\tau_g \geq \pi/(2\omega)$, where $\omega = E/\hbar$ is the Bohr frequency. See also [27] for the adiabatic version of the Margolus-Levitin theorem, and [28] for a lower bound on the amount of energy needed to carry out an elementary logical operation on a quantum computer, with a given accuracy and in a given time. In the case of periodic control fields one can understand **A1** as the result of the standard solution to the driven two-level atom problem, where the probability of a transition between ground and excited state is given by $(\Omega_R/\Omega_R')\sin^2(\Omega_R' t/2)$, where Ω_R is the Rabi frequency and $\Omega_R' = (\Omega_R + \delta^2)^{1/2}$, where δ is the detuning. This expression for the transition probability yields $\tau_g\Omega_R' = O(\pi)$.
(iii) **A2** is shown to be necessary in [6]. **A3** is stated clearly in [7] (see the discussion in Sections 2.10 and 10 there). These, and additional assumptions [such as constant fault rate (independent of number of qubits) and parallelism (to correct errors in all blocks simultaneously)] are explicitly listed, e.g., also in [29], Section II.
(iv) **A3** is usually related to the Markovian assumption, however both notions, the space-time correlations of errors and the Markovian property, need some comments and explanations. Using the convolutionless formalism in the theory of open systems (see, e.g., [30]) it is always possible to resolve the total superoperator $\Lambda(t)$ as

$$\Lambda(t) = \prod_{i=1}^{n} \Lambda_i U_i \qquad (1)$$

where U_i are ideal unitary superoperators (corresponding to quantum logic gates), and Λ_i are linear maps, not necessarily completely positive (CP) or even positive. If

Λ_i are CP then we can always realize them by coupling to an evironment which is "renewed each time step". This is the "Markovian condition" as formulated in [7] (section 2.10). However, complete positivity is not a necessary condition for QEC, which only requires a linear structure [31, 32]. To obtain the Threshold Theorem one needs the following bound on the probability [7][Eq. (2.6)]:

$$\Pr(\text{fault path with } k \text{ errors}) \leq c\eta^k(1-\eta)^{v-k}, \qquad (2)$$

where η is the probability of a single error, c is a certain constant independent of η, and v is the number of error locations in the circuit. This bound implies that the k-qubit errors should scale as $\sim \eta^k$, i.e., that in the decomposition of Λ_j into k-qubit superoperators $L_j(k)$

$$\|L_j(k)\| \leq c\eta^k . \qquad (3)$$

As discussed in [33] (within the Born approximation), the condition (3) can strictly be satisfied only for temporally *exponentially* decaying reservoir correlation functions, while for realistic reservoir models the temporal decay is generically powerlike. The decay of reservoir correlation functions (i.e., localization in time) translates into localization of errors in space due to the finite speed of error propagation. On the other hand it is widely believed that the Markovian model can be understood as arising, to an excellent approximation, from coupling to a reservoir which is not only renewed at each time step, but whose influence is independent of the actual Hamiltonian dynamics of the open system, and is localized in space (independent errors model) [34]. A large part of the present paper is devoted to a critical discussion of this claim.
(v) We note that the recent papers on FT-QEC theory [19, 20, 21] relax the (Markovian) assumption **A3**, but do make **A1** (implicitly) and **A2**. We comment on these papers in Section VI E.

III. REVIEW OF MARKOVIAN MASTER EQUATIONS

The field of derivations of the quantum Markovian master equation (MME) is strewn with pitfalls: it is in fact non-trivial to derive the MME in a fully consistent manner. There are essentially two types of fully rigorous approaches, known as the *singular coupling limit* (SCL) and the *weak coupling limit* (WCL), both of which we consider below. See, e.g., the books [35, 36] for more details, as well as the derivation in [37].

Consider a system and a reservoir (bath), with self Hamiltonians H_S^0 and H_R, interacting via the Hamiltonian $H_{SR} = \lambda S \otimes R$, where S (R) is a Hermitian system (reservoir) operator and λ is the coupling strength. A more general model of the form $H_{SR} = \sum_\alpha \lambda_\alpha S_\alpha \otimes R_\alpha$ can of course also be considered and results in the same qualitative conclusions. Thus the total Hamiltonian is

$$H = (H_S^0 + H_C(t)) \otimes I_R + I_S \otimes H_R + H_{SR}, \qquad (4)$$

where $H_C(t)$ describes control over the quantum device (system), and I is the identity operator.

The SCL and WCL derivations start from the expansion of the propagator Λ for the reduced, system-only dynamics,

$$\rho_S(t) = \Lambda(t,0)\rho_S(0), \qquad (5)$$

computed in the interaction picture with respect to the renormalized, *physical*, time-dependent Hamiltonian $H_S(t) = H_S + H_C(t)$, where

$$H_S = H_S^0 + \lambda^2 H_1^{\mathrm{corr}}(t) + \cdots . \qquad (6)$$

The renormalizing terms containing powers of λ are "Lamb-shift" corrections due to the interaction with the bath (see, e.g., [38]). The lowest order (Born) approximation with respect to the coupling constant λ yields H_1^{corr}, while the higher order terms ($\cdots$) require going beyond the Born approximation. Introducing a cumulant expansion for the propagator,

$$\Lambda(t,0) = \exp \sum_{n=1}^{\infty} [\lambda^n K^{(n)}(t)], \qquad (7)$$

one finds that $K^{(1)} = 0$. The Born approximation consists of terminating the cumulant expansion at $n = 2$, whence we denote $K^{(2)} \equiv K$:

$$\Lambda(t,0) = \exp[\lambda^2 K(t) + O(\lambda^3)]. \qquad (8)$$

One obtains

$$K(t)\rho_S = \int_0^t ds \int_0^t du F(s-u) S(s)\rho_S S(u)^\dagger \\ +(\text{similar terms}) \qquad (9)$$

as the first term in a cumulant expansion [37]. Here $F(s) = \mathrm{Tr}(\rho_R R(s)R)$ is the autocorrelation function, where ρ_R is the reservoir state and $R(s)$ is R in the H_R-interaction picture, and $S(u)$ is S in the interaction picture with respect to the physical Hamiltonian $H_S(t)$. The "similar terms" in Eq. (9) are of the form $\rho_S S(s)S(u)^\dagger$ and $S(s)S(u)^\dagger \rho_S$.

At first sight $K(t) \sim t^2$, and this is true for small times (Zeno effect [39]). The Markov approximation means that we can replace $K(t)$ by an expression that is linear in t, i.e.

$$K(t) \simeq \int_0^t \mathcal{L}(s)ds \qquad (10)$$

where $\mathcal{L}(t)$ is a time-dependent Lindblad generator. That the Lindblad generator can be time-dependent even after transforming back to the Schrödinger picture is important for our considerations below.

A. Singular Coupling Limit

The SCL approach we present in this subsection underlies the standard derivation of the MME that can be found in almost any text concerning the Markov approximation, though not always under the heading "SCL" (e.g., [40], p.8, Eq. (1.36)). The rigorous derivation of the SCL is briefly discussed (with references) in [35], pp.36-38. It is based on a rescaling of the bath and system-bath Hamiltonians, which physically makes sense in the high-temperature limit only. We will shortly see the emergence of this limit.

In essence, the "naive SCL-Markov approximation" is obtained by the ansatz $F(s) = a\delta(s)$ for the autocorrelation function, whence

$$L(s)\rho_S = aS(s)\rho_S S(s)^\dagger + (\text{similar terms}). \qquad (11)$$

As a consequence, return to the Schrödinger picture gives a MME with the dissipative part independent of the Hamiltonian:

$$\frac{d\rho_S}{dt} = -i[H_S(t), \rho_S] + \mathcal{L}\rho_S,$$
$$\mathcal{L}\rho_S \equiv -\frac{1}{2}\lambda^2 a[S, [S, \rho_S]] \qquad (12)$$

More precisely, we must consider the multi-time bath correlation functions $F(t_1, ..., t_n) := \mathrm{Tr}[\rho_R R(t_1)...R(t_n)] := \langle R(t_1)...R(t_n)\rangle$. Here $R(t) := \exp(iH_R t)R\exp(-iH_R t)$ are the bath operators in the interaction picture, $\rho_R = \exp(-\beta H_R)/Z$ (where $\beta = 1/kT$, $Z = \mathrm{Tr}[\exp(-H_R/kT)]$) is the bath thermal equilibrium state at temperature T, which is a stationary state of the reservoir, i.e., $[H_R, \rho_R] = 0$. The influence of the environment on the system is entirely encoded into the $\{F(t_1, ..., t_n)\}_{n=2}^{\infty}$.[2] Heuristically, the Markov approximation can be justified under the following conditions:

1. The lowest order correlation function,

$$F(t) = \langle R(s+t)R(s)\rangle = \int_{-\infty}^{\infty} G(\omega)e^{-i\omega t}d\omega, \qquad (13)$$

can be approximated by a Dirac delta function:[3]

$$F(t) \simeq \left(\int_{-\infty}^{\infty} F(s)ds\right)\delta(t) = G(0)\delta(t) \qquad (14)$$

(white-noise approximation). Eq. (13) defines the *spectral density* $G(\omega)$, which is a key object in the theory.

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[2] $F(t_1)$ is constant by stationarity. We reserve the notation $F(t)$ for $F(t_1, t_2) \equiv F(t_2 - t_1)$ below.

[3] Note that stationarity implies that $F(t)$ does not depend on s.

2. Higher order correlation functions exhibit a Gaussian-type behavior, i.e., can be estimated by sums of products of the lowest order ones, and then, by condition (14), decay sufficiently rapidly.

Let us now comment on the physical relevance of the white-noise approximation.

First, the condition (14) cannot be satisfied in general. For example, in the important case of linear coupling to a bosonic field (e.g., electromagnetic field, phonons in solid state), we have $G(0) = 0$, which means (by inverse Fourier transform) that $\int_{-\infty}^{+\infty} F(t)dt = 0$, and therefore $F(t)$ *cannot be well approximated by* $\delta(t)$.

Second, even for models with $G(0) > 0$ there exists a universal relation, the so-called Kubo-Martin-Schwinger (KMS) condition, $\langle R(t)R(0)\rangle = \langle R(0)R(t + i\beta)\rangle$, which is valid for all quantum systems at thermal equilibrium. This implies:

$$G(-\omega) = e^{-\beta\omega}G(\omega) . \qquad (15)$$

(See, e.g., [35][pp.90-91], [41][pp.176-177], or [36][p.137].) The fundamental importance of the KMS condition is captured by the fact that it is necessary in order for thermodynamics to hold. The KMS condition implies a strong asymmetry of the spectral density $G(\omega)$ for low T, where T is measured relative to the presence of kT energy scales in the bath, i.e., relative to the range where $G(\omega)$ is non-vanishing. The KMS condition is relevant to our discussion since we make the reasonably minimalistic assumption that the reservoir (not the QC) is in thermal equilibrium.[4]

Third, $G(\omega)$ need not be flat even at high T (indeed, the KMS condition only implies that $G(\omega)$ is symmetric at high T). For example, this is the case for the electromagnetic field and for phonons, for which at $T > 0$ one has $G(\omega) \sim \omega^3/(1 - e^{-\hbar\omega/kT})$ for $|\omega| \leq \omega_{\text{cut}}$, and $G(\omega) = 0$ for $|\omega| > \omega_{\text{cut}}$. One can see that for high T ($kT \gg \hbar\omega_{\text{cut}}$), $G(\omega) \sim kT\omega^2$ is symmetric. Here ω_{cut} is the Debye frequency in the case of phonons, while for the electromagnetic field ω_{cut} should tend to infinity in the renormalization procedure. A flat $G(\omega)$ means a structureless bath, while physical systems always have a non-trivial structure depending on relevant energy scales.[5]

Now let us return to the implications of the SCL assumptions for the problem of FT-QEC. In order to derive the SCL from first principles, one rescales $H_R \rightarrow H_R/\epsilon^2$, rescales $H_{SR} \rightarrow H_{SR}/\epsilon$, but keeps H_S and ρ_R fixed.[6] The idea of this rescaling is that it accelerates the reservoir's evolution (via $H_R \rightarrow H_R/\epsilon^2$) and hence produces faster decay of the reservoir correlations, $F(t)$. To see this, note that the rescaling $H_{SR} \rightarrow H_{SR}/\epsilon$ increases the amplitude $F(0)$ to $F(0)/\epsilon^2$ (proportional to H_{SR}^2), while keeping the strength of the noise $\int_{-\infty}^{+\infty} F(t)dt = G(0)$ fixed (as can be seen via a change of variables $t \rightarrow t/\epsilon^2$ in the integral). This implies a faster decay of $F(t)$. The rescaling procedure is specifically designed to yield the delta correlation [Eq. (14)] in the limit as $\epsilon \rightarrow 0$. Note that if ρ_R is at thermal equilibrium at temperature T with respect to H_R, then, since $\rho_R = \exp(-\beta H_R)/Z$ is fixed, it must be at thermal equilibrium with respect to H_R/ϵ^2 at the temperature $T/\epsilon^2 \rightarrow \infty$, whence our mention of the high temperature limit, above. Further note that H_S is not rescaled since the SCL is (artificially) designed to produce "white noise" on the natural time scale of system's evolution, which is given by H_S.

Another, equivalent way to understand the emergence of the high-T limit is the following: For the Markovian condition $F(t) \simeq a\delta(t)$ to hold the spectral density must be flat: $G(\omega) = $ const. However, this is possible only in the limit $T \rightarrow \infty$ of the KMS condition. More precisely, *the Markovian condition can hold only if* $kT \gg \omega$ *over the entire spectrum of the system's Bohr frequencies.* Strictly speaking, $G(\omega)$ is never constant. The variation of $G(\omega)$ happens over the "thermal memory" time $\tau_{\text{th}} := 1/kT$. In the infinite T limit we then recover the case of zero memory-time, i.e., Markovian dynamics. Physically, it is enough to assume that $G(\omega)$ is essentially constant over the interval $[-\omega_0, \omega_0]$ where $kT > \omega_0 \gg$ system's Bohr frequencies. I.e., system energy scales must be compared to $1/\tau_{\text{th}}$ and this leads to the important realization that *the Markovian approximation can be consistent with the KMS condition only in the high temperature regime* $kT \gg E$, *where E is the system energy scale.* As we argue below, this fact presents a seri-

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[4] One may challenge the notion that the bath must always be in thermal equilibrium. E.g., consider an atom in a microwave cavity, with the cavity electromagnetic field initially in thermal equilibrium. Now suppose the atom is driven and is coupled to the cavity electromagnetic field, which therefore is no longer in equilibrium. However, is the internal electromagnetic field the relevant environment, or is it the external one? Clearly, the electromagnetic field inside the cavity is not a reservoir but itself a part of the system. This is because: a) its spectrum is discrete, b) its coupling to the atom (close to resonance) is enhanced. The reason these considerations matter is because b) implies the strong coupling regime, hence failure of the initial state tensor product structure assumption, hence difficulties with the separation of the system from the reservoir (dressed atom picture); a) implies that $F(t)$ is (quasi)-periodic with short Poincaré recurrences, hence a strongly non-Markovian regime, and thus associated difficulties for Markovian FT-QEC. On the other hand the external electromagnetic field has a continuous spectrum and the state product structure is easily satisfied, hence qualifies as a reservoir. This example merely serves to illustrate accepted notions regarding the division into well defined system and bath; for most practical purposes a thermal equilibrium is the simplest and most relevant model of an environment, and FT-QEC theory must be applicable to this setting.

[5] It is interesting to note that even if we try to enforce a flat $G(\omega)$ by, e.g., choosing an appropriate form factor for the spin-boson system, the obtained model – the so-called "Ohmic case" – is mathematically and physically ill defined (see [42]).

[6] Note that because different Hamiltonians are rescaled differently, this rescaling procedure is *not* equivalent to a direct rescaling of the time variable (which is what is done in the WCL, below).

ous difficulty in the context of FT-QEC, the issue being essentially that the requirement of a constant supply of nearly pure and cold ancillas contradicts the high-T limit needed for the Markov approximation to hold.

B. Weak Coupling Limit

In the SCL approach above there was no restriction on the time-dependence of the system Hamiltonian. However, the price paid is the high-T limit. Moreover, while mathematically the SCL is rigorous in the scaling limit, it is inconsistent with thermodynamics except in the $T \to \infty$ limit. On the other hand, the derivation by Davies, in his seminal 1974 paper [43], is perhaps the only derivation of the MME that is entirely consistent from *both the mathematical and physical* points of view. The Davies approach is based not on the high-T limit, but rather on the physically plausible idea of weak coupling. This is natural and consistent with thermodynamics at all temperatures.

More specifically, Davies' derivation does not invoke a flatness condition on $G(\omega)$ but is, of course, still subject to the KMS condition. In the Davies approach the Markov approximation is a consequence of weak coupling (and hence slow dynamics of the system in the interaction picture), and time coarse-graining, which leads to cancellation of the non-Markovian oscillating terms. The price we pay is the invalidity of this approach for time-dependent Hamiltonians, except in the adiabatic case. We explain this important comment below. Hence, while the Davies approach does not require the high-T limit, it imposes severe restrictions on the speed of quantum gates.

In his rigorous derivation Davies replaced the heuristic condition (14) by the weaker

$$\int |F(t)| dt < \infty. \tag{16}$$

This condition avoids the difficulties originating from the singularity of the SCL condition (14), and preempts the corresponding problems with the high-T limit.[7,8] We now consider the cases of a constant, periodic, and arbitrarily time-dependent control Hamiltonian. The constant case

is the one originally treated by Davies [43], and extended in [44] to time-dependent Hamiltonians assuming a slow ("adiabatic") change on the dissipation time scale $\lambda^2 t$. The non-constant cases we study here have, as far as we know, not been published before in the general scientific literature.

1. WCL for Constant H_C: Summary of the Original Davies Derivation

We present a simplified version of the discussion of the Markov approximation in [37]. Denote by E_k the Bohr energies (eigenvalues of H_S), let $\omega \in \{\omega_{kl} = E_k - E_l\}_{k,l}$, and let S_ω be the discrete Fourier components of the interaction picture S, i.e.,

$$S(t) = \exp(iH_S t) S \exp(-iH_S t) = \sum_\omega S_\omega \exp(i\omega t), \tag{17}$$

where H_S is the renormalized (physical) system Hamiltonian: the sum of the "bare" H_S^0 and a Lamb shift term (bath induced), as in Eq. (6). Equivalently,

$$[H_S, S_\omega] = \omega S_\omega. \tag{18}$$

We remark that in the original Davies paper the Bohr energies and Eq. (18) are computed with respect to the bare Hamiltonian H_S^0. Here we use the physical Hamiltonian H_S in order to take into account the fact that the Lamb shift term, although formally proportional to λ^2, can be large or even infinite after cut-off removal.

Then, it follows from Eq. (9) that

$$K(t)\rho_S = \sum_{\omega,\omega'} S_\omega \rho_S S_{\omega'}^\dagger \int_0^t e^{i(\omega-\omega')u} du \int_{-u}^{t-u} F(\tau) e^{i\omega\tau} d\tau$$
$$+ \text{ (similar terms)}. \tag{19}$$

The weak coupling limit is next formally introduced by rescaling the time t to t/λ^2 (van Hove limit). This enables two crucial approximations, which are valid in the resulting large-t limit:

1. We replace[9]

$$\int_0^t e^{i(\omega-\omega')u} du \approx t\delta_{\omega\omega'}. \tag{20}$$

This makes sense for

$$t \gg \max\{1/(\omega - \omega')\}. \tag{21}$$

*This violates **A1**, expressed in terms of the Bohr frequencies.* We see here already the emergence of an adiabatic criterion for the validity of the Markov approximation.

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[7] In some sense the weak coupling limit is similar to the Central Limit Theorem (CLT) in probability, and condition (16) is analogous to a rough upper bound on the second moment in the CLT. If it is not satisfied then the noise may be not Gaussian in the weak coupling limit. The value of $\int |F(t)| dt$ itself does not provide any meaningful physical parameter and can depend on some regularization/cut-off parameters.

[8] One can go further and ask how generic the Markovian case is, in the sense that Eq. (16) is satisfied. In fact, typically $F(t)$ decays as $1/t^\alpha$ (e.g., for the vacuum bath $\alpha = 4$ [33]), which means that in some cases ($\alpha \leq 1$) Eq. (16) can be violated. For a systematic treatment of these non-Markovian effects see, e.g., [30].

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[9] In a more rigorous treatment the Cauchy principal value must be used, but the result is essentially the same [37].

2. We replace $\int_{-u}^{t-u} F(\tau)e^{i\omega\tau}d\tau$ by the Fourier transform:

$$\int_{-u}^{t-u} F(\tau)e^{i\omega\tau}d\tau \approx G(\omega) = \int_{-\infty}^{\infty} F(\tau)e^{i\omega\tau}d\tau. \quad (22)$$

The physical validity of the last approximation is usually ignored, though one can make the following argument: On the LHS of Eq. (22), for a given Bohr frequency ω the Fourier-like integral must sample the function $F(\tau)$ with sufficiently high accuracy so that the Fourier transform approximation will be valid. To this end one needs a time t such that: (i) $t \gg 1/\omega$. This is a weaker condition than the previous one $[t \gg \max\{1/(\omega - \omega')\}]$ which involves differences of Bohr frequencies. (ii) The time t must be also much longer than the time scale of the wildest variations of $F(\tau)$, which is typically [as may be checked for simple models of spectral densities $G(\omega)$] given by $1/\omega_{\text{cut}}$, where ω_{cut} is a high-frequency cutoff. When $\omega < \omega_{\text{cut}}$ (i) implies (ii). Therefore typically Eq. (20) is a stronger assumption than Eq. (22).

Applying the approximations (20) and (22), we obtain $K(t)\rho_S = t \sum_\omega S_\omega \rho_S S_\omega^\dagger G(\omega) + (\text{similar terms})$, and hence it follows from Eq. (10) that $\mathcal{L}(s) = \mathcal{L}$ is the Davies generator in the familiar Lindblad form:

$$\frac{d\rho_S}{dt} = -i[H_S, \rho_S] + \mathcal{L}\rho_S,$$
$$\mathcal{L}\rho_S \equiv \frac{1}{2}\lambda^2 \sum_\omega G(\omega)([S_\omega, \rho S_\omega^\dagger] + [S_\omega \rho, S_\omega^\dagger]) \quad (23)$$

Several remarks are in order:
(i) The absence of off-diagonal terms in Eq. (23), compared to Eq. (19), is the hallmark of the Markovian limit. Namely, the Davies derivation relies on the cancellation of the non-Markovian off-diagonal terms $\sum_{\omega \neq \omega'} S_\omega \rho_S S_{\omega'}^\dagger \int_0^t e^{i(\omega-\omega')u}du$. This time coarse-graining is possible due to integration over the fast-oscillating $\int_0^t e^{i(\omega-\omega')u}$ terms over a long timescale, i.e., over $t \gg \max\{1/(\omega - \omega')\}$ (see also [38]). As remarked above, this violates **A1**, expressed in terms of the Bohr frequencies.
(ii) It follows from Bochner's theorem applied to the Fourier transform definition of $G(\omega)$ that $G(\omega) \geq 0$ [35][p.90], [36][p.136]; this result is essential for the complete positivity of the Markovian master equation in the WCL.
(iii) Davies' derivation showed implicitly that the notion of "bath's correlation time" is not well-defined – Markovian behavior involves a rather complicated cooperation between system and bath dynamics. More specifically, the relations (23) and (18) together imply that the noise and H_S are strongly correlated. In other words, contrary to what is often done in phenomenological treatments, *one cannot combine arbitrary H_S's with given S_ω's.* This point is particularly relevant in the context of FT-QEC, where it is common to assume Markovian dynamics and apply arbitrary control Hamiltonians.

Davies did not consider time-dependent system Hamiltonians in [43], but it is possible to generalize his derivation to allow for slowly varying system Hamiltonians [37, 44, 45]. That is, whenever the time scale of the variation of $H_C(t)$ is much longer than the inverse of the typical Bohr frequency (of H_S), it is possible to add $H_C(t)$ to the system Hamiltonian in Eq. (23), necessitating at the same time this change also in Eq. (18). This is a type of adiabatic limit (indeed, the S_ω in Eq. (18) can be interpreted, with H_S replaced by $H_S + H_C(t)$, as being adiabatic eigenvectors of the superoperator $[H_S + H_C(t), \cdot]$). We note that an alternative approach to adiabaticity in open quantum systems was recently developed in Ref. [46]. This approach, while being very general, is more phenomenological in that it postulates a convolutionless master equation, and then derives corresponding adiabaticity conditions. Closer in spirit to the Davies derivation is another recent approach to adiabaticity in open systems, which assumes slow system variation together with weak system-bath coupling [47].

2. WCL for Periodic Driving: Floquet Analysis

Before considering the case of periodic H_C let us consider briefly once more the case of a constant Hamiltonian in the so-called covariant dissipation setting. Covariance is the commutation condition $\mathcal{H}\mathcal{L} = \mathcal{L}\mathcal{H}$ where $\mathcal{H} = [H_S, \cdot]$ is the super-operator constant Hamiltonian, and $\mathcal{L}$ is the Davies generator. Covariance is an abstract property which is automatically fulfilled for the Davies generator.[10] It is convenient since it implies factorization of the full propagator into Hamiltonian and dissipative parts. Markovian dynamics obtained in the WCL as discussed above takes the form

$$\frac{d\rho}{dt} = (-i\mathcal{H} + \mathcal{L})\rho , \ t \geq 0, \quad (24)$$

where the most general form of the Lindblad (or Davies) $\mathcal{L}$ satisfying Eq. (24) is

$$\mathcal{L}\rho = \frac{1}{2} \sum_{\{\omega\}, j} \left([V_j(\omega), \rho V_j(\omega)^\dagger] + [V_j(\omega)\rho, V_j(\omega)^\dagger]\right). \quad (25)$$

Here $\{\omega\} \equiv \text{Spectrum}(\mathcal{H})$, i.e., the Bohr frequencies (differences of eigenvalues of H), and

$$\mathcal{H}V_j(\omega) = \omega V_j(\omega) \quad (26)$$

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[10] This can be verified by directly computing $\mathcal{H}\mathcal{L}$ and making use of Eq. (18) and the relation $[A, BC] = [A, B]C + B[A, C]$ (for operators A, B and C). A more elegant way to see this is to consider $\mathcal{L}(t) = \exp(-it\mathcal{H})\mathcal{L}\exp(it\mathcal{H})$ and note that Eq. (18) implies that $S(t)$ and $S^\dagger(t)$ rotate in opposite directions. Hence $\mathcal{L}(t) = \mathcal{L}$, whence $d\mathcal{L}(t)/dt = 0$ gives the result.

[i.e., Eq. (18)]. The solution, i.e., the dynamical semi-group is

$$\rho(t) = \Lambda(t)\rho(0), \ \Lambda(t) = e^{-it\mathcal{H}}e^{t\mathcal{L}}. \tag{27}$$

Now consider a periodic control Hamiltonian with period Θ

$$H_C(t) = H_C(t + \Theta), \ \Omega = 2\pi/\Theta. \tag{28}$$

(Note that Ω is *not* the Rabi frequency, which throughout this paper we denote by Ω_R.) The situation is then very similar to the standard (time-independent H_C) WCL, but the set of "effective Bohr frequencies" (Floquet spectrum) ω is now larger and is of the form $\{\omega + q\Omega\}$, $q = 0, \pm 1, \dots$. Here ω are Bohr frequencies for the Floquet unitary [defined in Eq. (30) below], i.e., differences of eigenvalues ϵ_α of the Floquet unitary, rather than $\{\omega\} = \text{Spectrum}(\mathcal{H})$ as above. As this set of "effective Bohr frequencies" is discrete the WCL still works, but the final Davies generator is more complicated, as we now show.

Define the time-ordered unitary propagator

$$U(t, s) \equiv \mathcal{T} \exp\left(-i\int_s^t H_S(u)du\right), \ t \geq s \tag{29}$$

which satisfies the properties $U(s, t) \equiv U(t, s)^{-1} = U(t, s)^\dagger$, $U(t, s)U(s, u) = U(t, u)$, $U(t + \Theta, s + \Theta) = U(t, s)$, and $\frac{d}{dt}U(t, s) = -iH_S(t)U(t, s)$, $\frac{d}{dt}U(t, s)^\dagger = iU(t, s)^\dagger H_S(t)$. The *Floquet unitary operator* is

$$F(s) \equiv U(s + \Theta, s), \tag{30}$$

with corresponding super-operator action

$$\mathcal{F}(s)\rho \equiv F(s)\rho F(s)^\dagger, \tag{31}$$

and Floquet eigenvectors $|\phi_\alpha\rangle$ and eigenvalues (quasi-energies) ϵ_α satisfying[11]

$$F(0)|\phi_\alpha\rangle = e^{-i\epsilon_\alpha\Theta}|\phi_\alpha\rangle. \tag{32}$$

It follows from standard Floquet theory that

$$U(t, 0)|\phi_\alpha\rangle = e^{-it\epsilon_\alpha}\sum_{q \in \mathbf{Z}} e^{-itq\Omega}|\phi_\alpha(q)\rangle, \tag{33}$$

i.e., the set $\{|\phi_\alpha(q)\rangle\}$ is a complete basis. Therefore we have at most as many q's as the dimension of the Hilbert space. That the number of q's is finite is important for our considerations below.

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[11] Note that the Floquet Hamiltonian $H_S(t) - id/dt$ operates on a different Hilbert space than $F(0)$ (the space of periodic functions with values in the system's Hilbert space). But its eigenvalues coincide with ϵ_α from Eq. (32).

We call a Lindblad generator $\mathcal{L}$ a "covariant dissipative perturbation of $H_S(t)$" if

$$\mathcal{F}(0)\mathcal{L} = \mathcal{L}\mathcal{F}(0) \tag{34}$$

We will assume this property, similarly to the case of a constant Hamiltonian described above. In fact, covariance holds for a periodic $H_S(t)$ and also for the corresponding WCL Davies generator. One can then derive the *covariant master equation* (we sketch the derivation below):

$$\frac{d\rho}{dt} = (-i\mathcal{H}(t) + \mathcal{L}(t))\rho \ , \ t \geq 0, \tag{35}$$

[compare to Eq. (24)] where

$$\mathcal{L}(t) = \mathcal{U}(t, 0)\mathcal{L}\mathcal{U}(t, 0)^\dagger, \tag{36}$$

$$\frac{d}{dt}\mathcal{U}(t, s) = -i\mathcal{H}(t)\mathcal{U}(t, s), \tag{37}$$

and where the general form of $\mathcal{L}$ appearing in Eq. (36) is given by Eq. (25), with $V_j(\omega)$ now being eigenvectors of $\mathcal{F}(0)$,

$$\mathcal{F}(0)V_j(\omega) = e^{-i\omega\Theta}V_j(\omega), \tag{38}$$

rather than of $\mathcal{H}$, as in Eq. (26). Moreover, here $\{\omega\} \equiv \{\epsilon_\alpha - \epsilon_\beta\}$, where ϵ_α are quasi-energies (effective Bohr frequencies) of the Floquet operator, rather $\{\omega\} \equiv \text{Spectrum}(\mathcal{H})$ as we saw in the case of constant H_C.

The solution replacing Eq. (27) is

$$\rho(t) = \Lambda(t, s)\rho(s), \quad t \geq s$$
$$\Lambda(t, s) = \mathcal{T} \exp\left\{\int_s^t (-i\mathcal{H}(u) + \mathcal{L}(u))\, du\right\} \tag{39}$$

By direct computation one can prove the following properties:

$$\mathcal{L}(t + \Theta) = \mathcal{L}(t), \tag{40}$$
$$\mathcal{F}(s)\mathcal{L}(s)\mathcal{F}(s)^\dagger = \mathcal{L}(s), \tag{41}$$
$$\Lambda(t, s)\Lambda(s, u) = \Lambda(t, u) \text{ for } t \geq s \geq u, \tag{42}$$
$$\Lambda(t + \Theta, s + \Theta) = \Lambda(t, s), \tag{43}$$
$$\Lambda(t, s) = \mathcal{U}(t, s)e^{-(t-s)\mathcal{L}(s)}. \tag{44}$$

To derive the covariant master equation (35) one considers the standard picture of an open system $S + R$ with the total Hamiltonian

$$H_{SR}(t) = H_S^0(t) + H_R + \sum_k S_k \otimes R_k, \tag{45}$$

(we neglect the Lamb shift correction here; it can be included, changing $H_S^0(t)$ into the physical Hamiltonian $H_S(t)$, by a suitable renormalization procedure), stationary reservoir state ρ_R, $[H_R, \rho_R] = 0$, $\text{Tr}(\rho_R \cdot) \equiv \langle \cdot \rangle_R$,

$\langle R_k \rangle_R = 0$. Then, exactly following a Davies-like calculation using a Fourier decomposition of $S(t)$, now governed by a periodic Hamiltonian, and making in particular again the crucial assumption Eq. (21), which now reads

$$t \gg \max\{1/(\omega - \omega' + m\Omega)\}, \ m = 0, \pm 1, \pm 2, \ldots \quad (46)$$

with $|m|$ upper-bounded by the dimension of the Hilbert space [see remark after Eq. (33)], one obtains Eq. (35) in the Davies WCL. The explicit form of the generator is:

$$\mathcal{L}\rho = \frac{1}{2} \sum_{k,l} \sum_{q \in \mathbf{Z}} \sum_{\{\omega\}} \hat{R}_{kl}(\omega + q\Omega)\{[S_l(q,\omega)\rho, S_k(q,\omega)^\dagger]$$
$$+ \ [S_l(q,\omega), \rho S_k(q,\omega)^\dagger]\}. \quad (47)$$

Here $\{\omega\} \equiv \{\epsilon_\alpha - \epsilon_\beta\}$, the Floquet spectrum, and

$$\hat{R}_{kl}(x) = \int_{-\infty}^{\infty} e^{-itx} \langle R_k(t)R_l \rangle_R dt \quad (48)$$

$$S_k(q,\omega) = \sum_{p \in \mathbf{Z}} \sum_{\{\epsilon_\alpha - \epsilon_{\alpha'} = \omega\}} \langle \phi_\alpha(p+q)|S_k|\phi_{\alpha'}(p)\rangle|\phi_\alpha\rangle\langle\phi_{\alpha'}|. \quad (49)$$

$S_k(q,\omega)$ is the part of $S(t)$ which rotates with frequency $\omega + q\Omega$ and can be computed using Eq. (33). Note that by diagonalizing the matrices $\hat{R}_{kl}$ one can transform the generator $\mathcal{L}$ of Eq. (47) into the form of Eq. (25), which allows one to read off the operators $V_j(\omega)$ appearing there.

Now to some important comments:

– *Timescale analysis*: Note that for the periodic case the differences of "Bohr frequencies" may be of the order of $1/\Theta$. Hence we conclude from Eq. (46) that one must average over many periods Θ, i.e., require $t \gg \Theta$. This can be interpreted as a condition that "the environment must learn that the Hamiltonian is periodic". This is exactly analogous to the adiabaticity condition in the adiabatic case: $H(t)$ must be constant over many inverse Bohr frequencies to "be recognised" by the environment. The periodic WCL is also a coarse-grained time description with the additional time scale Θ. *Note that arbitrarily fast periodic driving (small Θ) is incompatible even with the kind of generalized, finitely localized MME derived here, since then differences of Bohr frequencies matter in Eq. (46)* (recall that $\max|m|$ is bounded by the – typically small – dimension of the system Hilbert space).

– *Where is the Rabi frequency?* Note the dependence of the operators $S_k(q,\omega)$ on the Floquet eigenvalue differences $\epsilon_\alpha - \epsilon_{\alpha'}$. The usual Rabi frequency, $\Omega_R = 2dE/\hbar$ (d is the dipole moment, E is the electric field amplitude) arises in the dipole approximation, which we have not made here. The usual Rabi frequency is replaced in our non-perturbative treatment (in the sense of no multipole expansion) by the difference of Floquet eigenvalues

$\epsilon_\alpha - \epsilon_{\alpha'}$ in Eq. (49).[12]

– *More on the Rabi frequency*: As we saw, the non-Markovian terms vanish because of the time coarse-grained description. To attain this, we must average over times $t \gg \max_{\omega,\omega'}\{1/(\omega - \omega')\}$, but must also keep in mind that the longest relevant scale for coarse-graining is given by the exponential decay time τ (a *derived* quantity), i.e., we must have $t < \tau$. The Rabi frequency Ω_R is a difference of two Bohr frequencies ω, ω': $\Omega_R = \omega - \omega'$. This implies that coarse-graining does not makes sense if $\Omega_R\tau \ll 1$ [since then $t < \tau \ll 1/\Omega_R = 1/(\omega - \omega')$, in contradiction to the fundamental requirement on t]. In physical terms this means that the width of the spectral line ($\gamma = 1/\tau$) is larger than the level splitting Ω_R (see, e.g., Fig. 2.5 (i),(ii) in [40] for an illustration in the case of the incoherent fluorescence spectrum) and therefore "the environment has no time to recognize the details of the spectrum". On the other hand, when $\Omega_R\tau \gg 1$ (not inconsistent with the WCL), Ω_R must appear in the generator, as appears from our treatment of the case of periodic driving in the WCL, above. Unfortunately there are examples in the literature where an MME is written down subject to $\Omega_R\tau \gg 1$ but Ω_R does not appear in the generator [e.g., Eq. (2.96) in [40], where $\Omega_R \sim 10^{10}$Hz and $\tau \sim 10^{-8}$s].

– *Quantum optics considerations*: The Markov approximation is commonly accepted as an excellent approximation in quantum optics; see, e.g., the discussion of resonance fluorescence in [40][Ch.2]. This is also the basis for substantial confidence in the possibility of FT-QEC in quantum optical systems, such as trapped ions [48] and atoms trapped in microwave cavities [49]. Such arguments are based on the relative flatness of the damping constants $\gamma(\omega)$ as a function of frequency. This argument is closely related to the notion of the flatness of the spectral density $G(\omega)$ in the SCL, since the damping constants are proportional to $G(\omega)$ [see Eq. (23)]. For example, below Eq. (2.95) in [40] the author argues that one can write down a Rabi frequency-independent MME for resonance fluorescence since $\gamma(\omega_A)$ and $\gamma(\omega_A \pm \Omega_R)$ (where ω_A is the Bohr frequency) differ by less than 0.01% at optical frequencies and reasonable laser intensities. However, this ignores the corrections due to the Rabi frequency to the operators $S_k(q,\omega)$ [Eq. (49)]. This disagreement can be traced to the question of at which point in the derivation it is safe to neglect Ω_R; in [40] this is done on the basis of the flatness of $\gamma(\omega)$ before "a lot of tedious algebra" [40][p.48], but our Floquet analysis shows that, in fact, one cannot neglect the Rabi fre-

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[12] One can see that such a term also arises in the usual dipole approximation by considering, e.g., Eq. (2.94) in [40]. The interaction picture raising and lowering operators $\sigma_\pm(t)$ (for a two-level atom driven by a classical field) there oscillate with three "Bohr frequencies" $\omega_A, \omega_A \pm \Omega$, where $\Omega = 2dE/\hbar$ denotes the usual Rabi frequency. Hence the Rabi frequency is a difference of two Bohr frequencies.

quency relative to the Bohr frequency. This is relevant for our general discussion since the "*finitely localized* MME" which is the outcome of the Floquet analysis (see next comment) actually exhibits a weak non-Markovian character. Such deviations are, of course, important for FT-QEC, even if the effects are small. We revisit this point in Section V below.

– *Are there any non-Markovian effects at work here?* It seems that one should accept the *generalized notion* of a quantum Markovian master equation as the one given by Eqs. (35), (36) and (25), i.e., a master equation with a possibly time-dependent Lindblad generator. In Davies' generalization to the time-dependent case [44] ("adiabatic WCL") the dissipative generator $\mathcal{L}(t)$ depends on the Hamiltonian at the *same time t*. This is a type of "*local* generalized MME". On the other hand, in the periodic WCL treated here, the dissipative generator $\mathcal{L}(t)$ depends on the Hamiltonians $H_S(u)$ from an interval, say $[0, t]$ $(t < \Theta)$, as can be seen from Eq. (36), which involves $\mathcal{U}(t, 0)$. This is therefore a type of "*finitely localized* MME", though one could argue that it exhibits a weakly non-Markovian character because of this dependence of the dissipative generator on the past. On the other hand, a non-Markovian master equation (in the convolutionless formalism [36]) is also given by Eq. (35), but the generator is *not* of Lindblad form [in particular, it is not of the form (32)], and may depend on the Hamiltonian in the *distant* past. The weight of distant past contributions depends on the decay properties of $F(t)$ which are, generically, not exponential but rather powerlike. In the WCL the non-Lindbladian terms vanish due to the oscillating character of the $e^{i(\omega - \omega')u}$ terms in Eq. (19).

– *The original Davies derivation*: We note that the Davies result is a limit theorem which states that for a sufficiently small coupling constant the WCL semigroup is a good approximation to the real dynamics. However, Davies' theorem itself does not provide the conditions under which a given physical coupling is "small enough". In particular, one cannot extract from Davies' theorem under what conditions the fast oscillating terms vanish. This can, however, be done by a more heuristic analysis, as done above.

3. WCL for an Arbitrary Pulse

We now consider the case

$$H_C(t) = H_0 + f(t)H_1, \tag{50}$$

i.e., an arbitrary driving field. This is, of course, the case of most interest in FT-QEC. It follows from Fourier analysis that this case can be treated qualitatively as a "superposition" of periodic perturbations discussed above. For a single frequency Ω the validity of the Markovian approximation is restricted by the condition (46): $t \gg \max\{1/(\omega - \omega' + m\Omega)\}$. The discreteness of the frequencies $\{\omega\}$ and $\{m\Omega\}$ is key: it allows for condition

(46) to be satisfied with finite t. A pulse $f(t)$ has a continuous band of frequencies of width $\Gamma \simeq 1/\tau_g$ (where τ_g is the gate duration), with amplitudes (Fourier transform) $\hat{f}(\Omega)$, which add to and smear the effective Bohr spectrum $\{\omega\}$. If the pulse is long (a slow gate) then only a narrow band appears, and the smearing effect is unimportant. More precisely, if $1/\tau_g$ is much smaller than the typical difference of the Bohr frequencies, the "energy quanta" $m\Omega$ [with $|m|$ restricted by the (typically small) dimension of the system Hilbert space] cannot fill the gap between ω and ω' and the condition (46) can be satisfied. This is our adiabatic approximation. For fast pulses, when $1/\tau_g$ is comparable to $|\omega - \omega'|$, the condition (46) cannot be fulfilled: the effective Bohr spectrum becomes quasi-continuous and the denominator in condition (46) becomes abitrarily small. The result is that the WCL analysis breaks down and non-Markovian effects dominate.

Thus, the condition for the adiabatic limit (Markov approximation valid) is: "the width of the band is much smaller than the minimal difference of the effective Bohr frequencies". This is in contradiction with the fast gate assumption, **A1**.

C. Section Summary

The main advantage of the MME (23) is its consistency with thermodynamics. Namely, as a consequence of the KMS condition (15) and the condition (18), for a generic initial state the system tends to its thermal equilibrium (Gibbs) state at the temperature of the heat bath [35]. (An important exception to this rule are states within a decoherence-free subspace [50, 51], but these states are not generic due to required symmetry properties of the system-bath interaction.) Therefore the dissipative part of the generator must depend strongly on the Hamiltonian dynamics. This is consistent with the notion of a coarse-grained description familiar from the study of MMEs: the bath needs a time much longer than $\max_{\omega_{kl}} 1/\omega_{kl}$ to "learn" the system's Hamiltonian in order to drive it to a proper Gibbs state. In other words, *the Markov approximation is, equivalently, a long-time limit* (compared to $\max_{\omega_{kl}} 1/\omega_{kl}$ – the system's Bohr frequencies), and one cannot expect this approximation to be valid at short times. However, FT-QEC assumes operations on a time-scale that is short on the scale set by $\max_{\omega_{kl}} 1/\omega_{kl}$.

Strictly speaking the MME (23) is valid only when H_S is not time dependent. As we have shown, one can relax this by assuming slowly varying H_S, giving rise to an "adiabatic MME", Eqs. (35), (36) and (25). However, to accept Eqs. (35), (36) and (25) as a genuine Markovian description is somewhat of a stretch, since the real question is not whether one obtains the Lindblad form, but rather *how $\mathcal{L}(t)$ depends on the Hamiltonians $H_S(u)$, locally (i.e. $u \simeq t$) or nonlocally. For fast gates and generic environments the dependence is non-local,*

involving memory effects. In any case, the crucial condition that must be satisfied for a (generalized) MME is Eq. (46), which implies that the average Bohr spectrum must be discrete. In essence, as long as the applied control does not spoil this discreteness a (generalized) MME can be derived. On the other hand, *this means that fast gates are incompatible with the MME,* in violation of **A1** of FT-QEC theory. The corollary: *finite speed of gates implies non-Markovian effects.*

IV. ARE THE STANDARD FT-QEC ASSUMPTIONS INTERNALLY CONSISTENT?

We now briefly summarize our examination of the assumptions of FT-QEC, in light of the considerations above, and highlight where there may be internal inconsistencies in FT-QEC. As discussed above, there are essentially two rigorous approaches to the derivation of the MME: (i) the SCL, which is compatible with arbitrarily fast Hamiltonian manipulations, but requires the high-T limit; (ii) the WCL, which is compatible with thermodynamics at arbitrary T, but requires adiabatic Hamiltonian manipulations.

The standard theory of FT-QEC (excluding Refs. [19, 20, 21]) requires a quantum computer (QC) undergoing Markovian dynamics, supplemented with a constant supply of cold and fresh ancillas. These assumptions are contradictory under the SCL, since the QC would have to be at high-T, while the ancillas require low-T on the same energy scale E (set by the Bohr energies of the system = computer + ancillas). Specifically, if we were to assume that for the ancillas too $kT \gg E$, they would quickly become highly mixed. If we insist that $kT \ll E$ for the ancillas, then by coupling them to the QC we can no longer assume, in the SCL, that the total system = QC + ancillas is described by Markovian dynamics.

If, on the other hand, we approach the problem from the (physically more consistent) WCL, then **A3** is incompatible with **A1** (the assumption of fast gates). Namely, in the WCL only adiabatic Hamiltonian manipulations are allowed. Specifically, the Markov approximation in the WCL requires a *discrete* system (effective) Bohr frequency spectrum, such that the condition $\tau_g \gg \max_{\omega_{kl}} 1/\omega_{kl}$ can be satisfied, hence violating the $\tau_g \omega_B = O(\pi)$ condition of **A1**. These conclusions are unavoidable if one accepts thermodynamics, since they follow from seeking a Markovian master equation that satisfies the KMS condition – a necessary condition for return to thermodynamic equilibrium in the absence of external driving. We take here the reasonable position that a fault tolerant QC cannot be in violation of thermodynamics.

V. POSSIBLE OBJECTIONS TO THE INCONSISTENCY

In this section we analyze a list of possible objections to the inconsistency we have pointed out.

A. Is thermodynamics relevant?

With respect to the SCL: "*Thermodynamics is irrelevant (since a QC need not ever be in thermal equilibrium).*"

Note that we never claim that the QC is in thermal equilibrium; only the bath is. This assumption is a simplification which allows us to use a single parameter T and therefore a single "thermal memory time" $\hbar/kT$. There is no reason to use a nonthermal bath or many heat baths with different temperatures: this does not make the spectral density flat and can only introduce more parameters.

B. Doesn't the interaction picture save the day?

With respect to the WCL: "*Suppose we have the following Hamiltonian in the Schrodinger picture: $H = H_S + H_C(t) + H_{SR} + H_R$ where $||H_S|| \gg ||H_C|| =$control Hamiltonian $\gg ||H_{SR}||$. Then in the interaction picture with respect to H_S the term H_C is dominant and hence can implement fast gates. However, in the original Schrödinger picture H_C is small and hence the adiabatic limit for the derivation of the MME is satisfied. Thus we have an example where we can have fast gates (in the interaction picture) and still the WCL can be satisfied so that the Markovian limit can be reached. Moreover, this is the relevant limit relevant for quantum optics, e.g., trapped ions.*"

There are a number of problems with this argument. First, one should be more careful about the formulation of the condition for adiabaticity. It can be stated as $|d\omega(t)/dt| \ll \omega(t)^2$, where $\omega(t)$ is a "relevant" Bohr frequency. Merely comparing norms as above does not guarantee adiabaticity. Second, in the quantum optics context we note the following. For three-level trapped ions we have two Bohr frequencies: a large, time-independent ω_1, and a small, time-dependent $\omega_2(t)$ (degenerate levels splitting). Only ω_2 is "relevant" because it is related to gates, and then the adiabatic condition implies that $|d\omega_2(t)/dt|$ is correspondingly small, which contradicts the fast gate condition **A1**. Third, the inequality $||H_C|| \gg ||H_{SR}||$ is in fact not satisfied in the Markovian WCL, where $||H_{SR}||$ diverges (one should not confuse the small system-reservoir coupling parameter involved in the van-Hove limit with the operator norm, which can be infinite).

C. Doesn't quantum optics provide a counterexample?

With respect to the WCL: "*Trapped ions and other quantum optics systems provide a counter-example: a system experimentally satisfying Markovian dynamics and allowing fast Rabi operations.*"

We have already addressed quantum optical systems in Section III B 2. Let us add a few comments. We do not know of any quantum optics experiment testing the Markov approximation with the accuracy relevant for FT-QEC (for quantum dots, on the other hand, non-Markovian effects are very visible). We know that for constant, and also for strictly periodic Hamiltonians (which corresponds in quantum optics to a constant external laser field), the Davies derivation can be applied (or extended, as in Section III B) and the Markov approximation is applicable. The problem appears for fast gates. It would be difficult to test the Markov approximation in this case with the required accuracy, because, e.g., the results depend on the shape of the pulse. A relevant example is resonance fluorescence, as described in [40][pp.43-61], and as discussed in Section III B 2. The damping effects are only present in the widths of spectral lines – see [40][p.61, Fig. 2.5]. The Markov approximation gives Lorentzians while non-Markovian dynamics may give rise to more complicated lineshapes. Consider a 2-level atom like in [40] Section 2.3.2., and in particular the final formula Eq. (2.96), which describes resonance fluorescence via a MME. The author claims that for typical parameters in quantum optics the dissipative part does not depend on the Rabi frequency Ω_R [recall our discussion in Section III B 2]. Hence, as the gates are entirely related to Ω_R, it appears that either fast or slow gates are possible. The argument is based on the small ratio $\Omega_R/\omega_A < 10^{10}/10^{15}$ (where ω_A is the Bohr frequency). This is fine for replacing the spectral density at $\omega_A \pm \Omega_R$ by the density at ω_A, but the subsequent argument that we can replace [in Eq. (2.94)] Ω_R by 0 is inaccurate. This would be correct only if the decay time $\tau = 1/\gamma$ is short enough such that $\Omega_R\tau \ll 1$. However, as explained in Section III B 2, in this case the Davies type averaging makes no sense physically. In fact, typically for radiation damping $\tau = 10^{-8}$s, and then $\Omega_R\tau < 100$ only. Hence for a fixed Ω_R we do in fact not have a simple Lindblad generator (of the type (2.96) in [40]), but rather a more complicated generator with Lindblad operators depending on the Rabi frequency, as in Eq. (47). Again, in the derivation of a proper generator an averaging over terms of the form $\exp(-i\Omega_R t)$ must be performed. Therefore the condition for the adiabatic approximation involves the Rabi frequency Ω_R and cannot be satisfied for fast gates. For experiments based on *spectral measurements* the difference between the two types of generators we have just discussed is probably irrelevant for many reasons; however, the quantum state of the atom at a given moment is sensitive to a small change in the Lindblad operators, and this is important in a fault

tolerant implementation of quantum logic gates.

D. Is A1 truly an assumption of FT-QEC?

With respect to the WCL: "*Doesn't **A1** impose an unnecessary constraint on FT-QEC, in that gates are not required to satisfy the condition $\tau_g\omega = O(\pi)$?*"

In other words, one might argue in favor of slow gates, where instead the condition is $\tau_g\omega \gg O(\pi)$. Such gates are certainly relevant in the context of the adiabatic quantum computing (AQC) paradigm [52], holonomic QC [53, 54], or topological quantum computing (TQC) [55, 56, 57]. We comment in more detail on AQC, HQC, and TQC in Section VI. The question of interest to us is whether an adiabatic gate satisfying $\tau_g\omega \gg O(\pi)$ is applicable to the standard FT-QEC paradigm we are considering here, and which is very different from AQC, HQC, and TQC.

First, let us clarify that by gates we mean one and two-qubit unitaries picked from well-known discrete and small sets of universal gates [58]. An algorithm is constructed via a sequence of such gates, and computational complexity is measured in terms of the minimal number of required gates. Of course one can instead join all gates used in a given algorithm into a single unitary and call this a gate, but then one runs into the problem of finding a relevant (physical) Hamiltonian and quantifying computational complexity. For a given gate there are infinitely many Hamiltonian realizations. Among these are fast ones (optimal) which satisfy $\tau_g\omega = O(\pi)$ and slow ones (adiabatic) satisfying $\tau_g\omega \gg O(\pi)$ (all inequalities here are in the sense of orders of magnitude). For example, consider a π-rotation. The fast (optimal) realization satisfies $\tau_g\omega = \pi$ (compatible with **A1**), while the slow (adiabatic) one satisfies $\tau_g\omega = \pi + 2\pi n$ with $n \gg 1$ (contradicts **A1**).

Now, one may ask whether a slow realization of gates can prevent the inconsistency with the WCL. We argue, based on computational complexity considerations, that the answer to this question is negative. To see this, note first that non-Markovian errors are uncorrectable in standard FT-QEC. Therefore such non-Markovian, uncorrectable errors accumulate during the computation (by definition, they are not corrected by "Markovian FT-QEC"), and in order to keep them under control, the probability of such errors per gate, $p_{\text{non}-M}$, should scale as

$$\begin{aligned} p_{\text{non}-M} &\sim O[1/(\text{volume of algorithm})] \\ &= O[1/(\text{input size})^\alpha], \end{aligned} \quad (51)$$

where α is some fixed power. Now, it follows from our discussion in Section III B that the more adiabatic the evolution, the smaller is the probability of the non-Markovian errors per gate. Therefore, if one writes the adiabaticity condition as $\tau_g\omega > M$, where $M \gg 1$ is the "adiabatic slowness parameter", then the probability of non-

Markovian errors should satisfy

$$p_{\text{non}-\text{M}} \sim O(1/M^{\beta}), \qquad (52)$$

where β is another fixed power [ω (the Bohr or Rabi frequency) is limited essentially by the choice of physical system]. Comparing the two expressions for $p_{\text{non}-\text{M}}$, we see that M must grow with input size. This means that if one works with adiabatic gates in order to keep the dynamics (approximately) Markovian, the result is that one must slow the gates in proportion to the input size (to some power). This, however, violates the threshold condition of FT-QEC, in which the input size and gate times are independent parameters (see, e.g., Theorem 12 in [7]).

E. Measurements

With respect to both the WCL and the SCL: "*Recent results on fault-tolerant QC using measurements only (e.g., [59, 60]) render all the claimed problems irrelevant.*"

Indeed, we have so far discussed only the problems with quantum logic gates. Moreover, measurements are an integral part of FT-QEC theory as well, in particular to reset and disentangle ancillas before they are introduced into an error-correction circuit. Therefore some remarks on the use of measurements are in order.

In the most advanced FT-QEC scheme of [7], measurements are performed at the end of the computation. However, this approach demands a high resource overhead, which may make it impractical. Therefore, more recent proposals (e.g., [17, 61]) rely on feedback mechanisms employing the results of quantum measurements. Those "measurements in the middle of computation" are treated for simplicity as certain von-Neumann projective measurements (but with efficiency $\ll 1$) satisfying a *repeatability condition*. The latter implies that the subsequent measurements reduce the measurement error exponentially as their number increases. This assumption should be carefully scrutinized, within realistic Hamiltonian models of quantum measurement treated as a dynamical process. Here, again one can expect that the tacit assumption of statistical independence of repeated measurements is in conflict with the non-Markovian character of the dynamics of open quantum systems.

As all proposed measurement schemes are based on electromagnetic interactions, it should be possible to construct a rather general Hamiltonian framework and apply it to various particular implementations. Indeed, this has been done, e.g., for a single-electron tunneling (SET) transistor coupled capacitively to a Josephson junction qubit [62]. Rather than assuming that the measurement apparatus is coupled to the system whenever measurements must be performed – an option which is hard to achieve in mesoscopic systems – Ref. [62] makes the reasonable assumption that the measurement apparatus is always coupled to the system, but is in a state of equilibrium when it is not needed. A measurement is then performed by driving the measuring device out of equilibrium, in a manner that dephases the qubit to be measured. Generic features emerging from this analysis are the existence of three different time-scales characterizing the measurement: the dephasing time, the measurement time (which may be longer than the dephasing time), and the mixing time (the time after which all the information about the initial quantum state is lost due to the transitions induced by the measurement). Ref. [62] thus arrives at a criterion for a "good" quantum measurement: the mixing time should be longer than the measurement time. A time-scale analysis of measurements in optical systems, accounting for spontaneous emission, can be found, e.g., in Ref. [63]. A fully consistent analysis of FT-QEC should account for the existence of such time-scales in a dynamic description of the measurement process. In particular, it is important to set appropriate bounds on these time-scales, so that they may be taken into account in a threshold calculation (an analysis based on a stochastic error model was reported in Ref. [14]).

F. Degenerate Qubits

With respect to the SCL: "*Degenerate qubits automatically satisfy the high T limit since their intrinsic energy scale vanishes.*"

Examples of degenerate qubits are common, e.g., in trapped ion quantum computing implementations where a pair of degenerate hyperfine states can serve as a qubit, with an auxiliary third level used to implement quantum logic gates via Raman transitions [64]. The case of degenerate qubits is somewhat more subtle to analyze within the context we have explained above. Naively, in such a case the high-T limit is indeed automatically satisfied, since the system energy scale is zero. Therefore it appears that one could claim that the SCL version of the Markov approximation is attainable. However, upon closer examination this still seems problematic. Indeed, the vanishing of an energy scale for degenerate qubits holds, strictly speaking, only for fully adiabatic techniques, e.g., HQC [53, 54]. Otherwise transformations between logical states are achieved by resorting to effective Hamiltonians which involve *virtual* transitions. For instance, if $|0\rangle$ and $|1\rangle$ denote degenerate qubit levels (e.g., hyperfine levels of an ion), one can introduce far-detuned (e.g., laser) couplings of $|0\rangle$ and $|1\rangle$ with a third auxiliary level. Second order perturbation theory then yields the effective Hamiltonian $H_{\text{eff}} = -(\Omega_{\text{R}}^2/\Delta)|1\rangle\langle 0|+\text{h.c.}$, where Ω_{R} and Δ are the laser Rabi coupling and detuning, respectively. Therefore we see that an effective, small but non-vanishing, energy scale $E_1 := \Omega_{\text{R}}^2/\Delta$ is introduced. (Note that in order for perturbation theory to be valid one must have $\Omega_{\text{R}} \ll \Delta$, which in turn implies $E_1 \ll \Delta$.) Yet another energy scale is provided by the spectral width E_2 of the laser pulse shape $\Omega_{\text{R}}(t)$; in order to suppress

unwanted *real* transitions, one must impose in addition that $E_2 \ll \Delta$. At any rate, the appearance of these new system-energy scales implies that once again the SCL-type contradiction applies. On the other hand, we can make both E_1 and E_2 small at the price of lengthening the gating time ($\tau_g \simeq \max\{1/E_1, 1/E_2\}$). This implies, once again, an adiabatic limit and the applicability of the WCL. Therefore it appears that as long as one restricts manipulations to adiabatic ones (thus contradicting **A1**), quantum computing with degenerate qubits is possible even in the Markovian limit. We expand on this viewpoint below.

G. Impure Ancillas

With respect to the SCL: "*Do ancillas really need to be pure?*"

What precisely is the role of the ancillas in QEC? A popular answer is that they serve as an "entropy sink" for the errors accumulated during the quantum computation. This entropy in the system arises from the entanglement between system and bath, and the role of the ancillas is to remove this entanglement. I.e., in a perfect quantum error correction step the entanglement between system and bath is transferred to the ancillas and bath. A natural objection to our SCL-based inconsistency is to claim that, in fact, ancillas need not be pure, or could perhaps even be highly mixed. However, this is not supported by the (current) standard theory of FT-QEC. Consider, e.g., an error correction circuit based on the Steane 7-qubit code. It takes as input ancillas prepared in the $|\psi\rangle_a = (|0_L\rangle + |1_L\rangle)/\sqrt{2}$ state, where $|0_L\rangle$ and $|1_L\rangle$ are codewords. The physical qubits which comprise such ancillas, are coupled bitwise via CNOT gates to the physical qubits making up the encoded data qubits in the circuit. If instead we input an ancilla in a mixed state, this is equivalent to inputting a classical mixture with erred codewords, e.g., $(1 - p)|\psi\rangle_a\langle\psi| + p|\phi\rangle_a\langle\phi|$, where $|\phi\rangle_a$ is an erred codeword. If one of these errors is a phase-flip, it feeds back (via the CNOT gates) into the data qubits, producing an error [65]. Without fault-tolerance this means that there are now two errors (in the ancillas block and the data block), which may be more than the code can handle. In FT-QEC theory such errors are accounted for, but their magnitude is bounded from above (e.g., p in the above example must be small). We note that an ancilla which is initially entangled with the data qubits (violating the assumption of being introduced into the circuit in a tensor-product state) is essentially equivalent to the case of an impure ancilla just described (tracing over the data qubits yields an impure ancilla state).

A more general approach showing the importance of the assumption of pure ancillas is the following (fairly standard account of QEC).

i) *Preparation.–*

Let the initial state of system + reservoir + ancillas,

with respective Hilbert spaces $\mathcal{H}_S, \mathcal{H}_R, \mathcal{H}_A$, be: $\rho^0_{SRA} = |\psi_S\rangle\langle\psi_S| \otimes |0_R\rangle\langle0_R| \otimes \rho_A$, where we have allowed for ancillas in a mixed state ρ_A.

ii) *System-reservoir interaction (decoherence).–*

$$\rho^0_{SRA} \xrightarrow{U_{SR}} \rho^1_{SRA} = \sum_{e,e' \in \mathcal{E}} U_e|\psi_S\rangle\langle\psi_S|U_e^\dagger \otimes |e_R\rangle\langle e'_R| \otimes \rho_A, \tag{53}$$

where e's denote the *errors* belonging to the set $\mathcal{E}$ that the code $\mathcal{C}$ can correct, and where $|e_R\rangle$ are the corresponding states of the reservoir. The error operators U_e are assumed to be unitary and with linear span of dimension $|\mathcal{E}|$.

iii) *System-ancilla interaction (syndrome extraction).–*

This interaction takes the form $U_{SA} = \sum_{e \in \mathcal{E}} \Pi_e \otimes T_e$ where the T_e's are unitaries over $\mathcal{H}_A$ such that $T_e|0_A\rangle = |e_A\rangle$ and $\Pi_e \cong I_{\mathcal{C}} \otimes |e\rangle\langle e|$ [13]:

$$\rho^1_{SRA} \xrightarrow{U_{SA}} \rho^2_{SRA}$$
$$= \sum_{e,e' \in \mathcal{E}} U_e|\psi_S\rangle\langle\psi_S|U_{e'}^\dagger \otimes |e_R\rangle\langle e'_R| \otimes T_e\rho_A T_{e'}. \tag{54}$$

iv) *Error recovery.—*

Unitary recovery is implemented via $\tilde{U}_{SA} = |\mathcal{E}|^{-1/2}\sum_{e \in \mathcal{E}} U_e^\dagger \otimes I_R \otimes |e_A\rangle\langle e_A|$, where for unitarity we need $\langle e_A|e'_A\rangle = \delta_{e,e'}$. By applying $\tilde{U}_{SA}$ and tracing over both R and A (assuming the $|e_R\rangle$'s too are orthonormal) one obtains

$$\rho^{\text{out}}_S = \frac{1}{|\mathcal{E}|} \sum_{e,f \in E} U_f^\dagger U_e|\psi_S\rangle\langle\psi_S|U_e^\dagger U_f \langle f_A|T_e\rho_A T_e^\dagger|f_A\rangle. \tag{55}$$

In the case of a pure ancillas $\rho_A = |0_A\rangle\langle0_A|$ one has $\langle f_A|T_e\rho_A T_e^\dagger|f_A\rangle = |\langle f_A|e_A\rangle|^2 = \delta_{f,e}$ and therefore the ideal case $\rho^{\text{out}}_A = |\psi_S\rangle\langle\psi_S|$ is recovered. One can also consider the fidelity

$$F := \langle\psi_S|\rho^{\text{out}}_S|\psi_S\rangle$$
$$= |\mathcal{E}|^{-1} \sum_{e,f \in E} |\langle\psi_S|U_f^\dagger U_e|\psi_S\rangle|^2 \langle f_A|T_e\rho_A T_e^\dagger|f_A\rangle. \tag{56}$$

Provided the error operators U_f satisfy the condition for a non-degenerate code $\langle\psi_S|U_f^\dagger U_e|\psi_S\rangle = \delta_{f,e}$ [31], one obtains $F = |\mathcal{E}|^{-1} \sum_{e \in \mathcal{E}} \langle e_A|T_e\rho_A T_e^\dagger|e_A\rangle = \langle 0_A|\rho_A|0_A\rangle$. Clearly, $F = 1$ iff $\rho_A = |0_A\rangle\langle0_A|$, i.e., *the ancillas are pure.* One can also consider non-unitary recovery via ancilla measurements and conditional unitaries, with Kraus operators given by $A_e = |\mathcal{E}|^{-1/2}U_e^\dagger \otimes I_R \otimes |e_A\rangle\langle e_A|$. The conclusion that the ancillas' state must be pure is unchanged.

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[13] We know that $\mathcal{H}_S \cong \mathcal{C} \otimes \mathcal{S} \oplus \mathcal{D}$ [$\mathcal{S}$=syndrome subsystem, $\dim\mathcal{D} = |\mathcal{E}|$; $\mathcal{D}$=remainder (=0 for subspace-based codes)] [31, 66, 67].

We note that FT is obtained by adding concatenation and, in steps iii) and iv), preparing and coupling encoded ancillas with the system in a suitable way, e.g., as in the Steane-code example above. In this case it is permissible to allow slightly impure ancillas, and relax the assumptions that, in step ii) the environment couples only to the system, and in steps iii,iv), the environment does not act. This formulation, however, does not allow arbitrarily mixed-state ancillas, as argued in the Steane-code example. While such a formulation of FT-QEC theory might still emerge (for example, by using algorithmic cooling techniques [68, 69], which, however, at present assume perfect gates), it does not appear possible at present to relax the assumption of cold ancillas.

H. Hot QC, cold ancillas, and fast QC-ancilla interactions in the SCL

With respect to the SCL: *"One can keep the ancillas coupled to a separate cold bath and then couple them for only a short time to the QC: what matters then is the T_1 timescale and that one can be very long compared to the required ancilla-QC coupling time"*.

Let us paraphrase this objection. If one can make H_{SA} (system-ancillas) very large then one could beat the rate of ancilla heating by strongly coupling the QC and ancillas. I.e., suppose one would like to bring the ancillas in from their cold reservoir to couple to the system, which is coupled to a hot reservoir as required for the SCL. The ancillas then heat up fast, but there is a timescale associated with this heating ("T_1"), which one wishes to beat. Now if one could make the system-ancilla coupling very strong then one could, presumably, use the ancillas (e.g. for syndrome extraction) faster than their heating rate, while they are still sufficiently pure for fault tolerance purposes.

The simplest argument against this objection is the following. In the setting of the objection, the QC is described by the SCL (high T) while the ancillas are described by the WCL (low T). Strong and fast coupling between the QC and the ancillas is unacceptable according to the WCL because it is fast (only adiabatic manipulations are allowed), and according to the SCL because it is strong ("strong" refers to the system's Hamiltonian part, while in the SCL this Hamiltonian is weak in comparison with the system-bath coupling).

However, one could go on to argue that the ancillas are a different species than the QC qubits, and in particular have a different intrinsic (less dense) energy scale, so that they are at low T on the scale set by the QC qubits. In this case both ancillas and QC are described by the SCL. Then the problem with the objection is the following: recall that in the SCL (see Section III A) one must rescale H_{SR} and H_{AR} as H_{SR}/ϵ and H_{AR}/ϵ respectively, where here R denotes the common reservoir the system and the ancillas are coupled to. The heating rate is proportional to the square of the coupling strength to the

reservoir, i.e., to $1/\epsilon^2$, and hence diverges in the SCL. Therefore to beat the ancilla heating process via fast manipulation of the system-ancilla coupling one would have to rescale H_{SA} at least by $1/\epsilon^2$, but this contradicts the SCL derivation, where in fact one must keep H_{SA} fixed while rescaling H_{SR}. The reason for this is that, in the SCL derivation, it is the system (now including the ancillas) that sets the timescale against which reservoir correlations must be accelerated.[14]

VI. ALTERNATIVES TO MARKOVIAN FT-QEC

A. Nature of the non-Markovian errors in the WCL

While we have pointed out that, in the WCL, the application of fast gates is likely to violate the conditions required for Markovian dynamics to persist, we have not been specific about the type of non-Markovian effects that will emerge. It is well known that FT-QEC is capable of dealing with errors that change due to the application of gates. Namely, assume (slightly) faulty gates correcting a specific error model described by a CP map Λ [recall Eq. (1)], are applied in sequence, $\Lambda U'_N \Lambda U'_{N-1} \cdots \Lambda U'_1$, and these gates are (in some appropriate norm) close to the ideal gates $\{U_i\}_{i=1}^N$, i.e., for all i, $||U'_i U_i^\dagger - I|| \ll 1$. Then by inserting $U_i^\dagger U_i$'s everywhere one obtains the new sequence $\Lambda_N U_N \Lambda_{N-1} U_{N-1} \cdots \Lambda_1 U_1$, where $\Lambda_i := \Lambda U'_i U_i^\dagger$, and FT-QEC is capable of dealing with such a (gate-modified) error model. However,

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[14] Let us also consider the issue from the perspective of thermodynamics. This is not really necessary, since the arguments above about the SCL are rigorous, but is interesting in its own right. First, we remark that error correction should really be made to work at the common lower (initial ancillas') temperature. Heating a part of a QC only to be closer to the Markovian limit is a suboptimal strategy, because it increases the strength of the noise and stimulates entropy production. Second, in standard FT-QEC heat (entropy) flows from the QC to the ancillas only, while in reality one should expect a flow in both directions and additionally an entropy production. To see this let us ignore for the moment the coupling of the QC to the bath, and consider ancillas coupled to a heat bath at temperature T. The ancillas can be kept pure by maintaining an energy gap $\gg kT$. Assume that the initial state of QC (C) and ancillas (A) is a product state $|\psi_C\rangle \otimes |\psi_A\rangle$. Switching on the interaction H_{CA} we induce an equilibration process (because the dynamics is Markovian) of $C + A$ towards the Gibbs state $\rho_{CA} = \exp(-H_{CA}/kT)/Z$, which is *entangled* (here for simplicity H_{CA} contains not only the interaction but is the total Hamiltonian of $C + A$). After a single step of error correction the total state of $C + A$ can be modeled by $(1-p)U|\psi_C\rangle \otimes |\psi_A\rangle U^\dagger + p\rho_{CA}$, where U is unitary and $0 < p \ll 1$. Then we switch off the interaction with the ancillas. Whatever we do next separately with C and A, we cannot eliminate the error due to the entanglement present in the term $p\rho_{CA}$. This type of incorrectable error accumulates and destroys FT-QEC. This is the back flow of entropy from the ancillas bath to the QC, mentioned above.

the non-Markovian effects that arise due to the application of fast gates in the WCL, will in general *not* be describable by a simple time-local modification such as $\Lambda \to \Lambda U_i' U_i^\dagger$. This can be worked out, e.g., using the methods of Ref. [30].

In order to formulate consistent alternatives to standard, Markovian FT-QEC theory, it seems useful to start with a Hamiltonian formulation. As the discussion below will illustrate, it appears that a hybrid approach will be necessary, which combines alternatives to standard QC with a new version of FT-QEC.

B. Adiabatic Quantum Computing (AQC)

We keep **A2** and **A3**, discard **A1** (fast gates), and work in a purely adiabatic mode, thus permitting a consistent WCL. This may indeed be possible using the adiabatic quantum computing (AQC) approach of Farhi et al. [52]. At present there is little understanding of the fault-tolerance of AQC. Some recent works explore AQC in the presence of decoherence and/or control errors [70, 71, 72, 73, 74, 75]. Indeed, the subject of the adiabatic approximation in open quantum systems has only very recently been addressed [46], and used to study AQC in open systems [74]. Error correcting codes for AQC were introduced very recently in [76], but the corresponding universal Hamiltonians involve many-body interactions (four and six-body for 1-local and 2-local errors, respectively).

C. Holonomic Quantum Computing (HQC)

Another possibility for keeping **A2** and **A3**, and discarding **A1**, is provided by HQC [53, 54]. HQC is an adiabatic scheme which relies on Abelian or non-Abelian geometric phases to implement quantum logic gates. Quantum information is encoded in a *degenerate* set of eigenstates of a Hamiltonian depending on a set of controllable parameters, e.g., external laser fields (recall our discussion of degenerate states above). When these are adiabatically driven along a suitable closed path, the initial quantum state is transformed by a non-trivial unitary transformation (holonomy) that is geometrical in nature. The key point is that the geometrical nature of the quantum holonomies is believed to render HQC inherently robust against certain kinds of errors. This alleged fault-tolerance has only recently been seriously begun to be examined [77]; the emerging picture is that, while stability against decoherence must still be assessed, HQC seems to exhibit a strong robustness against stochastic errors in the control process generating the required adiabatic loops [78]. Moreover, in the adiabatic limit of Markovian dynamics it has been show that the geometric phase of a single qubit coupled to a magnetic field is robust against both dephasing and spontaneous emission (but not against bit flips) [79]. Nevertheless, since deviations from strict adiabaticity are inevitable, and adiabaticity is particularly challenging to satisfy in open quantum systems [46], it is tempting to combine HQC with FT-QEC in order to address the performance of HQC in the presence of decoherence errors. Alternatively, we note that a hybrid approach that seems to be rather promising is the embedding of HQC within a DFS [80]. This amounts to realizing a set of universal quantum gates, acting on a DFS, by means of non-abelian quantum holonomies. This strategy brings together the "best of two worlds": the quantum decoherence avoidance virtues of DFSs and the fault-tolerance of the all-geometric holonomic control. It is possible that such an approach can be implemented for quantum information processing in, e.g., trapped ions and quantum dots.

D. Topological Quantum Computing (TQC)

A robust way of performing quantum computations is based on excitations with fractional statistics, since they have several fault-tolerant properties built in. This idea is known as topological quantum computing (TQC) [55, 56, 57]. Physical realizations of the simplest versions of TQC have been considered in the literature, using, e.g., rotating Bose-Einstein condensates [81] and superconducting circuits [82]. Let $\mathcal{C}$ denote the manifold of quantum codewords. Strikingly, in TQC, one can have a trivial Hamiltonian, e.g., $H|_{\mathcal{C}} = 0$, but nevertheless obtain non-trivial quantum evolution due to the existence of an underlying topological *global* structure (boundary conditions). Quantum encoding is typically performed in a properly designed degenerate ground state $\mathcal{C}$. This fact implies, for low enough temperature, an exponential suppression of errors on encoded quantum information due to thermal fluctuations. More importantly, topological features can render such a ground state stable against errors represented by *local* operators, namely error operators that do not involve a number of qubits of the order of the size of the system. For example, in the so-called toric codes [55, 83], qubits are encoded in the ground-state manifold of a lattice of interacting spins in such a way that degenerate ground states are mutually connected only via high powers (scaling linearly with lattice size) of local operators. Thus, here the fault-tolerance properties are already built-in at the *physical level*. However, while one can argue that topological encoding provides a stable and passive quantum memory, it is not self-correcting as in today's "effectively naturally fault-tolerant" classical architectures (see Ref. [84] for an eloquent exposition of this point). Moreover, it is important to realize that as far as we know, in its present state TQC still requires active intervention, in the form of FT-QEC, when one tries to compute fault-tolerantly. Indeed, Preskill writes in Ref. [85][p.62], "It is therefore implicit that the temperature is small enough compared to the energy gap of the model that thermally excited anyons are too rare to cause trouble, that the anyons are kept far enough apart from

one another that uncontrolled exchange of charge can be neglected, and in general that errors in the topological quantum computation are unimportant. If the error rate is small but not completely negligible, then the standard theory of quantum fault tolerance can be invoked to boost the accuracy of the simulation as needed". Ref. [29] takes this approach and explicitly lists **A2** and **A3** as necessary requirements for fault-tolerant TQC. In contrast, **A1** is definitely *not* required in TQC: one performs computations by adiabatically dragging quasiparticles around one another, and these operations must be slow relative to the gap between the ground state and the first excited state. The larger the gap the easier it is to satisfy this adiabaticity condition, so this requirements is compatible with the thermal suppression of errors mentioned above. In addition, TQC requires time-dependent controls to read out the encoded data (Ref. [86] shows that all measurements can be postponed until the readout of the final result of the computation). However, a fully Hamiltonian analysis of the fault-tolerance of such measurements is still lacking. Nevertheless, one could argue that the error rate in a topological quantum computer could be made arbitrarily small by increasing the system size and careful engineering, so that (similarly to today's self-correcting, fault-tolerant classical computers), one could ultimately perform TQC without any active intervention other than read-out of the encoded data. An interesting, recent development in this direction was reported in Ref. [84], which suggests that certain three-dimensional quantum spin-lattices might be self-correcting.

E.   Non-Markovian Quantum Computing

We keep **A1** and **A2** but discard **A3** (the Markov approximation) at least in part. This appears to be a reasonable approach in many cases, since the Markov approximation is clearly a highly idealized limit (though it does hold remarkably well in some optical systems and in liquid state NMR). Indeed, the degree of accuracy to which the Markov approximation must be satisfied has been quantified, e.g., by Steane in [61]: the probability of an uncorrectable (i.e., non-Markovian) error per gate must be $< 10^{-10}$ for a computation involving 10^9 gates (this probability must scale with the input size, as explained in Section V D). Alternative approaches to dealing with non-Markovian baths are therefore of interest. For example, the papers [19, 20, 21] present an extension of FT-QEC theory to a non-Markovian setting. We offer in this context the following observations:

1. An important ingredient carried over directly and without change from Markovian FT-QEC theory, is the crucial role of the fresh and nearly pure ancillas [19, 20, 21]. We believe that the detailed mechanism for introducing and discarding ancillas at specific times should be reconsidered within a fully Hamiltonian framework.

2. As recognized and discussed in [19], the important as-sumption of a small norm of the system-bath interaction Hamiltonian (e.g., Eq. (58) in Ref. [20]) is not satisfied for some standard models of open systems. For example, a linear coupling to a bosonic heat bath involves unbounded interaction Hamiltonians and a high-frequency cutoff. In general, the assumption of a small norm of the system-bath interaction Hamiltonian is much stricter than the WCL and is not satisfied for most standard models of reservoirs.

Another approach to fault-tolerance in a non-Markovian setting is the recently developed time-concatenated dynamical decoupling method [22] (see also [87] for a version of dynamical decoupling with bounded-strength controls). However, comment 2. above about the small norm of the system-bath interaction Hamiltonian applies here as well. Therefore more general methods are required to deal with the full scope of baths one can expect in quantum computing implementations. A promising possibility in this direction is to incorporate fault-tolerant dynamical decoupling in a feedback loop.

VII.   CONCLUSIONS

We have listed a set of minimal assumptions made in the theory of fault-tolerant quantum error correction (FT-QEC): 1) fast gates (on the timescale set by the inverse of the relevant Bohr or Rabi frequency), 2) a supply of fresh and nearly pure ancillas, 3) a Markovian bath.

We have also reviewed the only two known rigorous general limits leading to Markovian dynamics: the singular coupling limit (SCL), which involves taking a high temperature limit, and the weak coupling limit (WCL), which requires either a constant or an adiabatic system Hamiltonian, and averaging over long times in comparison with the inverse of the relevant Bohr frequency. These two limits allow one to replace the reservoir autocorrelation function by a Dirac delta, which leads to the Markovian limit.

A close examination of the assumptions of FT-QEC has led us to conclude that assumption 3 can be sustained together with assumption 1 in the SCL, and together with assumption 2 in the WCL. However, it is not possible to maintain all three assumptions in either the SCL or the WCL. We therefore conclude that, at present, there exists an inconsistency in the formulation of the theory of FT-QEC for Markovian baths. We have also listed a number of alternatives to Markovian FT-QEC which, from the point of view adopted here, are free of inconsistencies. However, none of these alternatives is so comprehensive as to include the full range of errors one might expect in a full-scale implementation of quantum computing. In particular, recent results on fault tolerance in non-Markovian settings [19, 20, 21, 22], while representing a significant step forward, make a crucial assumption about the smallness of the norm of the system-bath interaction Hamiltonian, which severely restricts the class of physical reservoirs.

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